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2010 J. Phys. A: Math. Theor. 43 055207

(http://iopscience.iop.org/1751-8121/43/5/055207)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **43** (2010) 055207 (14pp) doi:10.1088/1751-8113/43/5/055207

Painlevé VI and Hankel determinants for the generalized Jacobi weight

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Received 2 August 2009, in final form 10 December 2009
Published 14 January 2010
Online at stacks.iop.org/JPhysA/43/055207

Abstract

We study the Hankel determinant of the generalized Jacobi weight $(x-t)^{\gamma}x^{\alpha}(1-x)^{\beta}$ for $x \in [0,1]$ with $\alpha, \beta > 0$, $t < 0$ and $\gamma \in \mathbb{R}$. Based on the ladder operators for the corresponding monic orthogonal polynomials $P_n(x)$, it is shown that the logarithmic derivative of the Hankel determinant is characterized by a Jimbo–Miwa–Okamoto σ-form of the Painlevé VI system.

PACS numbers: 02.30.Gp, 02.10.Yn, 02.30.Hq

1. Introduction and statement of results

Let $P_n(x)$ be the monic polynomials of degree n in x and orthogonal with respect to the generalized Jacobi weight $w(x;t)$, that is,

$$\int_0^1 P_m(x)P_n(x)w(x)\,\mathrm{d}x = h_n\delta_{m,n}, \qquad h_n > 0, \quad m,n = 0,1,2,\ldots, \tag{1.1}$$

where

$$P_n(x) = x^n + \mathrm{p}_1(n)x^{n-1} + \cdots \tag{1.2}$$

and

$$w(x) := w(x;t) = (x-t)^{\gamma}x^{\alpha}(1-x)^{\beta}, \qquad x \in [0,1], \tag{1.3}$$

with $\alpha, \beta > 0$, $t < 0$ and $\gamma \in \mathbb{R}$. (In what follows, we often suppress the t-dependence for brevity. We believe that this will not lead to any confusion.) An immediate consequence of the orthogonality condition is the three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x), \tag{1.4}$$

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where the 'initial' conditions are taken to be $P_0(x) := 1$ and $\beta_0 P_{-1}(x) := 0$. Obviously, when $t \to 0^-$, these polynomials are reduced to the classical Jacobi polynomials up to some shift and rescaling, whose properties are well known; see [24]. In the literature, although people may use slightly different definitions, the generalized Jacobi polynomials have been studied from many points of view; for example, see [14, 15, 21, 25, 26]. In particular, Magnus [19] showed that an auxiliary quantity occurring in his study satisfies the Painlevé VI equation for certain parameters.

In this paper, we are concerned with the Hankel determinant for the generalized Jacobi weight

$$D_n(t) = \det\left(\int_0^1 x^{j+k}\, w(x;t)\,\mathrm{d}t\right)_{j,k=0}^{n-1} = \frac{1}{n!}\int_0^1 \cdots \int_0^1 \prod_{i<j}(x_i - x_j)^2 \prod_{k=1}^{n} w(x_k;t)\,\mathrm{d}x_k, \tag{1.5}$$

with $w(x;t)$ given in (1.3). Our motivation for this research mainly arises from the close relation between Hankel determinants and random matrix theory, which is of interest in mathematical physics. Indeed, the Hankel determinant defined in (1.5) can be viewed as the partition function for the unitary ensemble with eigenvalue distribution

$$\prod_{i<j}(x_i - x_j)^2 \prod_{k=1}^{n}(x_k - t)^{\gamma} x_k^{\alpha}(1 - x_k)^{\beta}\,\mathrm{d}x_k; \tag{1.6}$$

see the definitive book of Mehta [20] for a discussion of this topic. The main purpose of this work is to study the properties of $D_n(t)$ as a function of t. More precisely, we are going to show that the logarithmic derivative of Hankel determinant $D_n(t)$ is characterized by a Jimbo–Miwa–Okamoto σ-form of the Painlevé VI system. The appearance of Painlevé VI may not be so surprising somehow. As a matter of fact, it was already known that, for some special weight functions, the corresponding Hankel determinants are connected to the well-known nonlinear ordinary differential equations—Painlevé equations. In particular, it is first shown in [16] that a gap probability in the Jacobi polynomial ensemble is related to Painlevé VI, see also [7, 13] for further discussion. For the present general Jacobi case, although it is natural to 'guess' the existence of such a link, the precise form and specific quantity to which it is related to, however, is not clear.

Our approach is based on the ladder operator for orthogonal polynomials, which has been successfully applied to many other polynomials before; see [2, 3, 8, 10–12]. The main result is the following.

Theorem 1.1. *Let H_n be the logarithmic derivative of the Hankel determinant with respect to t,*

$$H_n(t) := t(t-1)\frac{\mathrm{d}}{\mathrm{d}t}\ln D_n(t) \tag{1.7}$$

and denoted by

$$\widetilde{H}_n := H_n + d_1 t + d_2, \tag{1.8}$$

with

$$d_1 = -n(n+\alpha+\beta+\gamma) - \frac{(\alpha+\beta)^2}{4},$$

$$d_2 = \frac{1}{4}[2n(n+\alpha+\beta+\gamma) + \beta(\alpha+\beta) - \gamma(\alpha-\beta)].$$

Then $\widetilde{H}_n$ satisfies the following Jimbo–Miwa–Okamoto σ-form of Painlevé VI in [18, 22]

$$\widetilde{H}_n'(t(t-1)\widetilde{H}_n'')^2 + \left\{2\widetilde{H}_n'(t\widetilde{H}_n' - \widetilde{H}_n) - \widetilde{H}_n'^2 - \nu_1\nu_2\nu_3\nu_4\right\}^2 = \left(\widetilde{H}_n' + \nu_1^2\right)\left(\widetilde{H}_n' + \nu_2^2\right)\left(\widetilde{H}_n' + \nu_3^2\right)\left(\widetilde{H}_n' + \nu_4^2\right), \tag{1.9}$$

with

$$\nu_1 = \frac{\alpha+\beta}{2}, \qquad \nu_2 = \frac{\beta-\alpha}{2}, \qquad \nu_3 = \frac{2n+\alpha+\beta}{2}, \qquad \nu_4 = \frac{2n+\alpha+\beta+2\gamma}{2}.$$

Remark 1.1. Due to the symmetric form of (1.9), the choice of ν_1, ν_2, ν_3 and ν_4 is not unique.

Remark 1.2. Although we assume $t < 0$ in the definition (1.3) of the weight function $w(x)$, theorem 1.1 also holds for $t > 1$ if $(x-t)^\gamma$ is substituted by $(t-x)^\gamma$. One may expect this theorem is valid for all real $t \neq 0, 1$ when $(x-t)^\gamma$ is replaced by $|x-t|^\gamma$, which is similar to what has been studied by Chen and Feigin [9]. Unfortunately, we cannot prove it at this moment.

Remark 1.3. If $\gamma = 0$, then $(x-t)^\gamma \equiv 1$ and we can readily reduce the polynomials $P_n(x)$ in (1.1) to the classical Jacobi polynomials, which are of course t independent. As a consequence, the formula (1.8) provides a trivial solution for the associated σ-form in (1.9). Moreover, if $\gamma = 1$, the Hankel determinant $D_n(t)$ defined in (1.5) is actually a polynomial in t of degree n and orthogonal with respect to the 'shifted' Jacobi weight $t^\alpha(1-t)^\beta$ on [0, 1]. By the classical theory of Jacobi polynomials (cf [24]), $D_n(t)$ satisfies the following second-order differential equation:

$$t(1-t)D_n'' - [(2+\alpha+\beta)t - \alpha - 1]D_n' + n(n+\alpha+\beta+1)D_n = 0. \tag{1.10}$$

Hence, if we denote by $u(t) := \frac{d}{dt}\ln D_n(t)$, it is easily seen that $u(t)$ is a solution of the following Riccati equation:

$$t(1-t)u' = t(t-1)u^2 + [(2+\alpha+\beta)t - \alpha - 1]u - n(n+\alpha+\beta+1) \tag{1.11}$$

for $t < 0$ in this special case. In addition, one can verify that $t(t-1)u(t) + d_1 t + d_2$ is a rational solution to the associated σ-form in (1.9).

From (1.4), it is easily seen that $\beta_n = h_n/h_{n-1}$. Since

$$D_n(t) = \prod_{j=0}^{n-1} h_j \tag{1.12}$$

(see equation (2.1.6) in [17]), we can express β_n in terms of the Hankel determinant as follows:

$$\beta_n = \frac{D_{n-1}D_{n+1}}{D_n^2}. \tag{1.13}$$

Therefore, it is also expected that there exists a certain relation between β_n and the Painlevé VI equation. Indeed, as a by-product of our main theorem, we find a first-order differential equation for β_n, whose coefficient is closely related to the Painlevé VI equation.

Theorem 1.2. *The recurrence coefficient β_n satisfies a first-order differential equation as follows:*

$$t\frac{d}{dt}\beta_n = (2 + R_{n-1} - R_n)\beta_n, \tag{1.14}$$

where R_n is related to the Painlevé VI equation in the following way. Let

$$W_n(t) := \frac{(t-1)R_n(t)}{2n+\alpha+\beta+\gamma+1} + 1. \tag{1.15}$$

Then $W_n(t)$ satisfies the Painlevé VI equation

$$W_n'' = \frac{1}{2}\left(\frac{1}{W_n} + \frac{1}{W_n - 1} + \frac{1}{W_n - t}\right)(W_n')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{W_n - t}\right)W_n' + \frac{W_n(W_n - 1)(W_n - t)}{t^2(t-1)^2}\left(\mu_1 + \frac{\mu_2 t}{W_n^2} + \frac{\mu_3(t-1)}{(W_n - 1)^2} + \frac{\mu_4 t(t-1)}{(W_n - t)^2}\right), \tag{1.16}$$

with

$$\mu_1 = \frac{(2n + \alpha + \beta + \gamma + 1)^2}{2}, \qquad \mu_2 = -\frac{\alpha^2}{2}, \qquad \mu_3 = \frac{\beta^2}{2}, \qquad \mu_4 = \frac{1 - \gamma^2}{2}.$$

The present paper is organized as follows. In section 2, we give a brief introduction to the ladder operator theory and state three compatibility conditions S_1, S_2 and S_2'. Based on these supplementary conditions, we introduce some auxiliary constants in section 3. Their relations with other quantities such as the coefficient of orthogonal polynomials, Hankel determinant, etc are also derived for further use. We conclude this paper with the proof of theorems 1.1 and 1.2 in sections 4 and 5, respectively.

2. Ladder operators and compatibility conditions

The ladder operators for orthogonal polynomials have been derived by many authors with a long history, we refer to [4–6, 11, 23] and references therein for a quick guide. Following the general set-up (see for example [11]), we have the lowering and raising ladder operator for our generalized Jacobi polynomials $P_n(z)$:

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} + B_n(z)\right)P_n(z) = \beta_n A_n(z) P_{n-1}(z), \tag{2.1}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} - B_n(z) - \mathsf{v}'(z)\right)P_{n-1}(z) = -A_{n-1}(z) P_n(z), \tag{2.2}$$

with $\mathsf{v}(z) := -\ln w(z)$ and

$$A_n(z) := \frac{1}{h_n}\int_0^1 \frac{\mathsf{v}'(z) - \mathsf{v}'(y)}{z - y}[P_n(y)]^2 w(y)\,\mathrm{d}y, \tag{2.3}$$

$$B_n(z) := \frac{1}{h_{n-1}}\int_0^1 \frac{\mathsf{v}'(z) - \mathsf{v}'(y)}{z - y}P_{n-1}(y)P_n(y)w(y)\,\mathrm{d}y. \tag{2.4}$$

Note that $A_n(z)$ and $B_n(z)$ are not independent but satisfy the following supplementary conditions.

Proposition 2.1. *The functions $A_n(z)$ and $B_n(z)$ defined in (2.3) and (2.4) satisfy the following compatibility conditions:*

$$B_{n+1}(z) + B_n(z) = (z - \alpha_n)A_n(z) - \mathsf{v}'(z), \tag{S_1}$$

$$1 + (z - \alpha_n)[B_{n+1}(z) - B_n(z)] = \beta_{n+1}A_{n+1}(z) - \beta_n A_{n-1}(z). \tag{S_2}$$

Proof. Using the recurrence relation and the Christoffel–Darboux formulas, all the formulas (2.1), (2.2), (S_1) and (S_2) can be derived by direct calculations. We refer to [4–6, 23] for details. Also, see [10, 11] for a recent proof. □

From (S_1) and (S_2), we can derive another identity involving $\sum_{j=0}^{n-1} A_j$ which is very helpful in our subsequent analysis. We state the result in the following proposition.

Proposition 2.2.

$$B_n^2(z) + \mathsf{v}'(z)B_n(z) + \sum_{j=0}^{n-1} A_j(z) = \beta_n A_n(z)A_{n-1}(z). \qquad (S_2')$$

Proof. See the proof of theorem 2.2 in [8]. □

The conditions S_1, S_2 and S_2' are usually called the compatibility conditions for the ladder operators, which will play an important role in our future analysis. Although the author obtained an equivalent form of S_2' in [19], he did not study it further. We would also like to emphasize that, as in [8], the condition S_2 is essential in the present case (see remark 3.1 below), while in [2, 3], only the conditions S_1 and S_2' are sufficient to derive all the relations.

3. The analysis of the ladder operators

3.1. Some auxiliary constants

To prove our results stated in section 1, we would like to introduce some auxiliary constants first. For the weight function $w(z)$ given in (1.3), we know

$$\mathsf{v}(z) := -\ln w(z) = -\alpha \ln z - \beta \ln(1-z) - \gamma \ln(z-t). \qquad (3.1)$$

Hence,

$$\mathsf{v}'(z) = -\frac{\alpha}{z} - \frac{\beta}{z-1} - \frac{\gamma}{z-t} \qquad (3.2)$$

and

$$\frac{\mathsf{v}'(z)-\mathsf{v}'(y)}{z-y} = \frac{\alpha}{zy} + \frac{\beta}{(z-1)(y-1)} + \frac{\gamma}{(z-t)(y-t)}. \qquad (3.3)$$

Since the right-hand side of the above formula is rational in z, it is easily seen that both $A_n(z)$ and $B_n(z)$ are also rational in z from their definitions in (2.3) and (2.4). More precisely, we have the following lemma.

Lemma 3.1. *We have*

$$A_n(z) = \frac{R_n^*}{z} - \frac{R_n}{z-1} + \frac{R_n - R_n^*}{z-t}, \qquad (3.4)$$

$$B_n(z) = \frac{r_n^*}{z} - \frac{r_n}{z-1} + \frac{r_n - r_n^* - n}{z-t}, \qquad (3.5)$$

where

$$R_n^* := \frac{\alpha}{h_n}\int_0^1 [P_n(y)]^2 w(y)\frac{\mathrm{d}y}{y}, \qquad (3.6)$$

$$R_n := \frac{\beta}{h_n}\int_0^1 [P_n(y)]^2 w(y)\frac{\mathrm{d}y}{1-y}, \qquad (3.7)$$

$$r_n^* := \frac{\alpha}{h_{n-1}}\int_0^1 P_{n-1}(y)P_n(y)w(y)\frac{\mathrm{d}y}{y}, \qquad (3.8)$$

$$r_n := \frac{\beta}{h_{n-1}} \int_0^1 P_{n-1}(y) P_n(y) w(y) \frac{\mathrm{d}y}{1-y}. \tag{3.9}$$

Proof. Inserting (3.3) into (2.3) gives us

$$A_n(z) = \frac{1}{h_n}\left[\frac{\alpha}{z}\int_0^1 [P_n(y)]^2 w(y)\frac{\mathrm{d}y}{y} + \frac{\beta}{z-1}\int_0^1 [P_n(y)]^2 w(y)\frac{\mathrm{d}y}{y-1} + \frac{\gamma}{z-t}\int_0^1 [P_n(y)]^2 w(y)\frac{\mathrm{d}y}{y-t}\right]. \tag{3.10}$$

Applying integration by parts, we obtain

$$\int_0^1 [P_n(y)]^2 w(y) \mathrm{v}'(y)\,\mathrm{d}y = -\int_0^1 [P_n(y)]^2\,\mathrm{d}w(y) = \int_0^1 2P_n'(y)P_n(y)w(y)\,\mathrm{d}y = 0. \tag{3.11}$$

On account of (3.2) and the above formula, we have

$$\gamma\int_0^1 [P_n(y)]^2 w(y)\frac{\mathrm{d}y}{y-t} = \beta\int_0^1 [P_n(y)]^2 w(y)\frac{\mathrm{d}y}{1-y} - \alpha\int_0^1 [P_n(y)]^2 w(y)\frac{\mathrm{d}y}{y}. \tag{3.12}$$

A combination of (3.10) and (3.12) yields (3.4).

In a similar manner, we get (3.5) from (2.4). In that case, we need to make use of the following equality:

$$\gamma\int_0^1 P_{n-1}(y)P_n(y)w(y)\frac{\mathrm{d}y}{y-t} = -nh_{n-1} - \alpha\int_0^1 P_{n-1}(y)P_n(y)w(y)\frac{\mathrm{d}y}{y} + \beta\int_0^1 P_{n-1}(y)P_n(y)w(y)\frac{\mathrm{d}y}{1-y}. \tag{3.13}$$

□

In view of the compatibility conditions (S_1), (S_2) and (S_2'), one can derive the following relations among the four auxiliary quantities R_n, R_n^*, r_n, r_n^*.

Proposition 3.1. *From (S_1), we obtain the following equations:*

$$r_{n+1}^* + r_n^* = \alpha - \alpha_n R_n^*, \tag{3.14}$$

$$r_{n+1} + r_n = (1-\alpha_n)R_n - \beta, \tag{3.15}$$

$$tR_n^* - (t-1)R_n = 2n + 1 + \alpha + \beta + \gamma, \tag{3.16}$$

where the constants R_n, R_n^*, r_n and r_n^* are defined in (3.6), (3.7), (3.8) and (3.9), respectively.

Proof. Substituting (3.4) and (3.5) into (S_1), we have

$$B_{n+1}(z) + B_n(z) = \frac{r_{n+1}^* + r_n^*}{z} - \frac{r_{n+1}+r_n}{z-1} + \frac{r_{n+1}+r_n - r_{n+1}^* - r_n^* - 2n - 1}{z-t} \tag{3.17}$$

and

$$(z-\alpha_n)A_n(z) - \mathrm{v}'(z) = (z-\alpha_n)\left[\frac{R_n^*}{z} - \frac{R_n}{z-1} + \frac{R_n - R_n^*}{z-t}\right] + \frac{\alpha}{z} + \frac{\beta}{z-1} + \frac{\gamma}{z-t}$$
$$= \frac{\alpha - \alpha_n R_n^*}{z} - \frac{(1-\alpha_n)R_n - \beta}{z-1} + \frac{(t-\alpha_n)(R_n - R_n^*) + \gamma}{z-t}. \tag{3.18}$$

Comparing the coefficients at $O(z^{-1})$, $O((z-1)^{-1})$ and $O((z-t)^{-1})$ in the above two formulas, we get

$$r_{n+1}^* + r_n^* = \alpha - \alpha_n R_n^*, \tag{3.19}$$

$$r_{n+1} + r_n = (1 - \alpha_n) R_n - \beta, \tag{3.20}$$

$$r_{n+1} + r_n - r_{n+1}^* - r_n^* - 2n - 1 = (t - \alpha_n)(R_n - R_n^*) + \gamma. \tag{3.21}$$

A combination of the above three formulas gives our proposition. □

Proposition 3.2. *From (S_2'), we have the following equations:*

$$(r_n^*)^2 - \alpha r_n^* = \beta_n R_n^* R_{n-1}^*, \tag{3.22}$$

$$r_n^2 + \beta r_n = \beta_n R_n R_{n-1}, \tag{3.23}$$

$$(2n + \beta + \gamma) r_n - (2n + \alpha + \gamma) r_n^* + 2 r_n r_n^* - n(n + \gamma) = \beta_n (R_{n-1} R_n^* + R_n R_{n-1}^*) \tag{3.24}$$

and

$$\sum_{j=0}^{n-1} R_j = (2n + \alpha + \beta + \gamma)(r_n - r_n^*) - n(n + \gamma) + \frac{(2n + \alpha + \beta + \gamma) r_n^* + n(n + \beta + \gamma)}{1 - t}, \tag{3.25}$$

where the constants R_n, R_n^*, r_n and r_n^* are defined in (3.6), (3.7), (3.8) and (3.9), respectively.

Proof. Again we substitute (3.4) and (3.5) into (S_2') to obtain

$$\begin{aligned} B_n^2(z) + \mathsf{v}'(z) B_n(z) + \sum_{j=0}^{n-1} A_j(z) = {} & \frac{(r_n^*)^2 - \alpha r_n^*}{z^2} + \frac{r_n^2 + \beta r_n}{(z-1)^2} + \frac{(r_n - r_n^* - n)(r_n - r_n^* - n - \gamma)}{(z-t)^2} \\ & + \frac{\alpha r_n - \beta r_n^* - 2 r_n r_n^*}{z(z-1)} + \frac{(r_n - r_n^* - n)(2 r_n^* - \alpha) - \gamma r_n^*}{z(z-t)} \\ & - \frac{(r_n - r_n^* - n)(2 r_n + \beta) - \gamma r_n}{(z-1)(z-t)} + \sum_{j=0}^{n-1} \left[\frac{R_j^*}{z} - \frac{R_j}{z-1} + \frac{R_j - R_j^*}{z-t} \right] \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} \beta_n A_n(z) A_{n-1}(z) = {} & \frac{\beta_n R_n^* R_{n-1}^*}{z^2} + \frac{\beta_n R_n R_{n-1}}{(z-1)^2} + \frac{\beta_n (R_n - R_n^*)(R_{n-1} - R_{n-1}^*)}{(z-t)^2} \\ & - \frac{\beta_n (R_n^* R_{n-1} + R_n R_{n-1}^*)}{z(z-1)} + \frac{\beta_n (R_n^* R_{n-1} - 2 R_n^* R_{n-1}^* + R_n R_{n-1}^*)}{z(z-t)} \\ & + \frac{\beta_n (R_n R_{n-1}^* - 2 R_n R_{n-1} + R_{n-1} R_n^*)}{(z-1)(z-t)}. \end{aligned} \tag{3.27}$$

Equating the coefficients of the above two formulas at $O(z^{-2})$, $O((z-1)^{-2})$ and $O((z-t)^{-2})$, we get (3.22), (3.23) and

$$(r_n - r_n^* - n)(r_n - r_n^* - n - \gamma) = \beta_n (R_n - R_n^*)(R_{n-1} - R_{n-1}^*), \tag{3.28}$$

respectively. A substitution of (3.22) and (3.23) into the above formula gives us (3.24). At $O((z-1)^{-1})$, note that

$$\frac{1}{z} = 1 + O(z-1), \qquad \frac{1}{z-t} = \frac{1}{1-t} + O(z-1), \qquad \text{as} \quad z \to 1.$$

It then follows from (3.26) and (3.27) that

$$-2r_nr_n^* - \beta r_n^* + \alpha r_n - \sum_{j=0}^{n-1} R_j - \frac{(r_n - r_n^* - n)(2r_n + \beta) - \gamma r_n}{1-t}$$
$$= -\beta_n \left[R_nR_{n-1}^* + R_{n-1}R_n^* + \frac{-R_nR_{n-1}^* + 2R_nR_{n-1} - R_{n-1}R_n^*}{1-t} \right]. \tag{3.29}$$

Combining (3.24) and the above formula, we have

$$\sum_{j=0}^{n-1} R_j = (2n + \alpha + \beta + \gamma)(r_n - r_n^*) - n(n+\gamma)$$
$$+ \frac{2\beta_nR_nR_{n-1} + (2n + \alpha + \beta + \gamma)r_n^* + n(n + \beta + \gamma) - 2r_n^2 - 2\beta r_n}{1-t}. \tag{3.30}$$

Eliminating $\beta_nR_nR_{n-1}$ in (3.30) with the aid of (3.23), we finally obtain (3.25). □

Remark 3.1. From another condition (S_2), we get one more equation as follows:

$$(t-1)(r_{n+1} - r_n) - t(r_{n+1}^* - r_n^*) - t + \alpha_n = 0. \tag{3.31}$$

Rewriting the above formula yields

$$\alpha_n = t(r_{n+1}^* - r_n^*) - (t-1)(r_{n+1} - r_n) + t. \tag{3.32}$$

Since it follows from (1.2) and (1.4) that

$$\alpha_n = \mathsf{p}_1(n) - \mathsf{p}_1(n+1), \qquad n = 0, 1, 2, \ldots, \tag{3.33}$$

with $\mathsf{p}_1(0) := 0$; hence, it is easily seen

$$-\sum_{j=0}^{n-1} \alpha_j = \mathsf{p}_1(n). \tag{3.34}$$

Inserting (3.32) into the above formula, we obtain

$$\mathsf{p}_1(n) = (t-1)r_n - tr_n^* - nt, \tag{3.35}$$

where we have made use of the initial conditions $r_0(t) = r_0^*(t) := 0$.

3.2. The recurrence coefficients

Not only the coefficients $A_n(z)$ and $B_n(z)$ of the ladder operators in (2.1) and (2.2) but also the recurrence coefficients α_n and β_n in (1.4) can be written in terms of the auxiliary quantities R_n, r_n and r_n^*. Here we do not need R_n^* since it is related to R_n in a simple way; see (3.16).

Lemma 3.2. *The recurrence coefficients α_n and β_n are expressed in terms of R_n, r_n and r_n^* as follows:*

$$(2n + 2 + \alpha + \beta + \gamma)\alpha_n = 2(t-1)r_n - 2tr_n^* + (1-t)R_n + (\alpha + \beta + 1)t - \beta \tag{3.36}$$

and

$$(2n - 1 + \alpha + \beta + \gamma)(2n + 1 + \alpha + \beta + \gamma)\beta_n = [tr_n^* - (t-1)r_n]^2$$
$$- (t-1)(2nt + \gamma t + \beta)r_n + t[(t-1)(2n+\gamma) - \alpha]r_n^* + n(n+\gamma)(t^2 - t). \tag{3.37}$$

Proof. We use (3.14) and (3.15) to eliminate r_{n+1}^* and r_{n+1} in (3.31) and get

$$[1 + tR_n^* - (t-1)R_n]\alpha_n = t(\alpha - 2r_n^*) - (t-1)(R_n - \beta - 2r_n) + t. \tag{3.38}$$

Substituting (3.16) into the above formula immediately gives us (3.36).

To derive the formula for β_n, we need to consider the identities in proposition 3.2. Multiplying both sides of (3.22) by t^2 and eliminating $t^2 R_n^* R_{n-1}^*$ with the aid of (3.16), we have

$$t^2((r_n^*)^2 - \alpha r_n^*) = (t-1)\beta_n[(t-1)R_{n-1}R_n + c_{n-1}R_n + c_n R_{n-1}] + c_{n-1}c_n\beta_n, \tag{3.39}$$

where $c_n := 2n+1+\alpha+\beta+\gamma$. Similarly, we multiply both sides of (3.24) by t and use (3.16) again to get

$$\begin{aligned} &t[(2n+\beta+\gamma)r_n - (2n+\alpha+\gamma)r_n^* + 2r_n r_n^* - n(n+\gamma)] \\ &\quad = \beta_n[2(t-1)R_{n-1}R_n + c_{n-1}R_n + c_n R_{n-1}]. \end{aligned}$$

On account of (3.23), it is readily derived from the above formula that

$$\begin{aligned} &t[(2n+\beta+\gamma)r_n - (2n+\alpha+\gamma)r_n^* + 2r_n r_n^* - n(n+\gamma)] - (t-1)\left(r_n^2 + \beta r_n\right) \\ &\quad = \beta_n[(t-1)R_{n-1}R_n + c_{n-1}R_n + c_n R_{n-1}]. \end{aligned} \tag{3.40}$$

Substituting (3.40) into (3.39) yields (3.37). □

Remark 3.2. For $n = 0$, from (1.4) and the definitions of $R_0(t)$, $r_0(t)$ and $r_0^*(t)$, it follows that

$$\begin{aligned} \alpha_0(t) &= \frac{(\alpha+1)\,{}_2F_1\left(\alpha+2, -\gamma, \alpha+\beta+3; \frac{1}{t}\right)}{(\alpha+\beta+2)\,{}_2F_1\left(\alpha+1, -\gamma, \alpha+\beta+2; \frac{1}{t}\right)}, \\ R_0(t) &= \frac{(\alpha+\beta+1)\,{}_2F_1\left(\alpha+1, -\gamma, \alpha+\beta+1; \frac{1}{t}\right)}{{}_2F_1\left(\alpha+1, -\gamma, \alpha+\beta+2; \frac{1}{t}\right)}, \\ r_0(t) &= r_0^*(t) = 0, \end{aligned}$$

where ${}_2F_1$ is the hypergeometric function; see [1, p 556]. The validity of (3.36) at $n = 0$ can be verified directly from the above formulas.

Furthermore, it is easily seen that

$$R_0(t) = \alpha+\beta+1+O(1/t), \tag{3.41}$$

as $t \to -\infty$.

3.3. The t dependance

Recall that our weight function depends on t; therefore, all of the quantities considered in this paper such as the coefficient of generalized Jacobi polynomials, Hankel determinant, etc can be viewed as functions in t. In this subsection, we will investigate their dependance with respect to this parameter. We start with the study the coefficient $\mathsf{p}_1(n)$ in (1.2).

Lemma 3.3. *We have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{p}_1(n) = r_n - r_n^* - n. \tag{3.42}$$

Proof. From the orthogonal property (1.1), we know

$$\int_0^1 P_n(x)P_{n-1}(x)w(x;t)\,\mathrm{d}x = 0.$$

Note that $P_n(x)$ is also dependent on t. Taking derivative of the above formula with respect to t gives us

$$\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}P_n(x)\;P_{n-1}(x)w(x;t)\,\mathrm{d}x + \int_0^1 P_n(x)P_{n-1}(x)\frac{\mathrm{d}}{\mathrm{d}t}w(x;t)\,\mathrm{d}x = 0.$$

It then follows from (1.1) to (1.3) that

$$h_{n-1}\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{p}_1(n) - \gamma\int_0^1 P_n(x)P_{n-1}(x)w(x)\frac{\mathrm{d}x}{x-t} = 0.$$

Combining (3.8), (3.9) and (3.13), we get (3.42) immediately. □

By (3.35) and the above lemma, it is easily seen that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{p}_1(n) = r_n - r_n^* - n = r_n + (t-1)\frac{\mathrm{d}}{\mathrm{d}t}r_n - r_n^* - t\frac{\mathrm{d}}{\mathrm{d}t}r_n^* - n. \tag{3.43}$$

Hence, we obtain the following nice relation between the derivatives of r_n and r_n^*:

$$t\frac{\mathrm{d}}{\mathrm{d}t}r_n^* = (t-1)\frac{\mathrm{d}}{\mathrm{d}t}r_n. \tag{3.44}$$

Next, we derive the following property about the Hankel determinant $D_n(t)$.

Lemma 3.4. *We have*

$$t\frac{\mathrm{d}}{\mathrm{d}t}\ln D_n(t) = n(n+\alpha+\beta+\gamma) - \sum_{j=0}^{n-1} R_j, \tag{3.45}$$

where R_j is defined in (3.7).

Proof. Differentiating (1.1) with respect to t yields

$$h_n' = -\gamma\int_0^1 [P_n(x)]^2 w(x)\frac{\mathrm{d}x}{x-t}. \tag{3.46}$$

This, together with (3.6), (3.7) and (3.12) implies

$$h_n' = h_n(R_n^* - R_n). \tag{3.47}$$

Using (3.16) to replace R_n^* by R_n, we find

$$t\frac{\mathrm{d}}{\mathrm{d}t}\ln h_n = 2n+1+\alpha+\beta+\gamma - R_n. \tag{3.48}$$

Then, a combination of (3.48) and (1.12) gives us (3.45). □

Finally, we derive differential equations for the recurrence coefficients α_n and β_n. They are the non-standard Toda equations.

Lemma 3.5. *The recurrence coefficients α_n and β_n satisfy the following differential equations:*

$$t\frac{\mathrm{d}}{\mathrm{d}t}\alpha_n = \alpha_n + r_n - r_{n+1}, \tag{T_1}$$

$$t\frac{\mathrm{d}}{\mathrm{d}t}\beta_n = (2 + R_{n-1} - R_n)\beta_n, \tag{T_2}$$

where R_n and r_n are defined in (3.7) and (3.9), respectively.

Proof. Applying $t\frac{\mathrm{d}}{\mathrm{d}t}$ to both sides of (3.33), we have from (3.42)

$$t\frac{\mathrm{d}}{\mathrm{d}t}\alpha_n = t(r_n - r_n^* - n) - t(r_{n+1} - r_{n+1}^* - n - 1).$$

(T_1) then follows from (3.31) and the above formula. Using (3.48), it is readily seen that

$$t\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{h_n}{h_{n-1}} = 2 + R_{n-1} - R_n.$$

Note that $\beta_n = h_n/h_{n-1}$, one easily gets (T_2) from the above formula. □

4. Proof of theorem 1.1

Now we are ready to prove our main theorem. The idea is to make use of lemma 3.4 to express r_n^* and r_n in terms of H_n and its derivative with respect to t. Then we derive two independent formulas for R_n in terms of r_n^* and r_n with the aid of (3.37) and (T_2). Equating these two formulas, finally we obtain an equation involving H_n, H_n' and H_n''. We first need the following proposition for r_n and r_n^*.

Proposition 4.1.

$$r_n^* = -\frac{n(n+\beta+\gamma)+(t-1)H_n'-H_n}{2n+\alpha+\beta+\gamma}, \tag{4.1}$$

$$r_n = \frac{n(n+\alpha+\gamma)-tH_n'+H_n}{2n+\alpha+\beta+\gamma}. \tag{4.2}$$

Proof. Recalling the definition of H_n in (1.7), we substitute (3.25) into (3.45) and get

$$H_n = [n(2n+\alpha+\beta+2\gamma)-(2n+\alpha+\beta+\gamma)(r_n-r_n^*)](t-1) \\ +(2n+\alpha+\beta+\gamma)r_n^*+n(n+\beta+\gamma). \tag{4.3}$$

Taking a derivative of the above formula with respect to t, it then follows from (3.44) that

$$(t-1)H_n'-H_n = -n(n+\beta+\gamma)-(2n+\alpha+\beta+\gamma)r_n^*, \tag{4.4}$$

which gives us (4.1). Eliminating r_n^* from the above two formulas, we get (4.2). □

Next we derive a proposition for R_n as follows.

Proposition 4.2. *The auxiliary quantity R_n has the following representations:*

$$R_n(t) = \frac{(2n+1+\alpha+\beta+\gamma)[l(r_n,r_n^*,t)-t(1-t)r_n'(t)]}{2k(r_n,r_n^*,t)}, \tag{4.5}$$

$$\frac{1}{R_n(t)} = \frac{l(r_n,r_n^*,t)+t(1-t)r_n'(t)}{2(2n+1+\alpha+\beta+\gamma)(\beta+r_n)r_n}, \tag{4.6}$$

where

$$l(r_n,r_n^*,t) := 2(1-t)r_n^2+[(2n-\beta+\gamma)t+2\beta+2tr_n^*]r_n \\ -(2n+\alpha+\gamma)tr_n^*-n(n+\gamma)t \tag{4.7}$$

and

$$k(r_n,r_n^*,t) := [tr_n^*-(t-1)r_n]^2-(t-1)(2nt+\gamma t+\beta)r_n \\ +t[(t-1)(2n+\gamma)-\alpha]r_n^*+n(n+\gamma)(t^2-t). \tag{4.8}$$

Proof. Using (3.23), we eliminate R_{n-1} in (3.40) and obtain

$$(2n+1+\alpha+\beta+\gamma)\frac{r_n^2+\beta r_n}{R_n}+(2n-1+\alpha+\beta+\gamma)\beta_n R_n \\ = t[(2n+\beta+\gamma)r_n-(2n+\alpha+\gamma)r_n^*+2r_nr_n^*-n(n+\gamma)]-2(t-1)(r_n^2+\beta r_n).$$

Replacing β_n in the above formula with the aid of (3.37), we have

$$\frac{2n+1+\alpha+\beta+\gamma}{R_n}\left(r_n^2+\beta r_n\right)+\frac{k(r_n,r_n^*,t)}{2n+1+\alpha+\beta+\gamma}R_n$$
$$=2(1-t)r_n^2+[(2n-\beta+\gamma)t+2\beta+2r_n^*t]r_n$$
$$-(2n+\alpha+\gamma)tr_n^*-n(n+\gamma)t. \qquad (4.9)$$

On the other hand, by applying $t\frac{\mathrm{d}}{\mathrm{d}t}$ to (3.37), it follows

$$(2n-1+\alpha+\beta+\gamma)(2n+1+\alpha+\beta+\gamma)t\frac{\mathrm{d}}{\mathrm{d}t}\beta_n$$
$$=t\Big[2t(r_n^*)^2+2(1-2t)r_nr_n^*+2(t-1)r_n^2+(2n-\beta+\gamma-(4n+2\gamma)t)r_n$$
$$-(2n+\alpha+\gamma-(4n+2\gamma)t)r_n^*-(t-1)(2nt+\gamma t+\beta)\frac{\mathrm{d}}{\mathrm{d}t}r_n$$
$$+t((t-1)(2n+\gamma)-\alpha)\frac{\mathrm{d}}{\mathrm{d}t}r_n^*+n(n+\gamma)(2t-1)\Big],$$

where we have used (3.44). A further substitution of (3.23), (3.37) and (T_2) into the above formula yields

$$\frac{(2n-1+\alpha+\beta+\gamma)(2n+1+\alpha+\beta+\gamma)}{R_n}\left(r_n^2+\beta r_n\right)-k(r_n,r_n^*,t)R_n$$
$$=2(t-1)r_n^2-[(2n-\beta+\gamma)t+2\beta+2r_n^*t]r_n+(2n+\alpha+\gamma)tr_n^*$$
$$+n(n+\gamma)t+t(1-t)(2n+\alpha+\beta+\gamma)\frac{\mathrm{d}}{\mathrm{d}t}r_n. \qquad (4.10)$$

Formulas (4.5) and (4.6) now follow from solving for R_n and $1/R_n$ from (4.9) and (4.10). □

Now we are ready to finish the proof of theorem 1.1.

Proof of theorem 1.1. Multiplying (4.5) and (4.6) gives us

$$t^2(t-1)^2[r_n'(t)]^2=l^2(r_n,r_n^*,t)-4k(r_n,r_n^*,t)(\beta+r_n)r_n, \qquad (4.11)$$

where $l(r_n,r_n^*,t)$ and $k(r_n,r_n^*,t)$ are given in (4.7) and (4.8), respectively. Recall that r_n^* and r_n can be written in terms of H_n and H_n', see (4.1) and (4.2). Therefore, the above formula actually gives us a nonlinear differential equation for H_n. Using (1.8) to replace H_n by $\widetilde{H}_n$, we finally get (1.9), which completes the proof of our theorem. □

5. Proof of theorem 1.2

We conclude this paper with the proof of theorem 1.2.

Proof of theorem 1.2. Firstly, we try to express r_n^* in terms of r_n, r_n', R_n and R_n'. To achieve this, we substitute (3.36) into (T_1) and get an equation involving r_n, r_n', r_n^*, $r_n^{*\prime}$, R_n' and r_{n+1}. Then we use (3.15) and (3.44) to eliminate r_{n+1} and $r_n^{*\prime}$. At the end, we arrive at

$$r_n^*=\frac{1}{2}+\frac{1}{2R_n}((t-1)R_n'-2r_n-(\alpha+\beta+1))+\frac{1}{2tR_n}((2n+\alpha+\beta+\gamma+2)(2r_n-R_n+\beta)$$
$$+(R_n+1)[2(t-1)r_n-(t-1)R_n+(\alpha+\beta+1)t-\beta]). \qquad (5.1)$$

Next, we insert the above formula into (4.5) and (4.6) and obtain a pair of *linear* equations in r_n and r_n'. Solving this linear system gives us

$$r_n=F(R_n,R_n') \qquad \text{and} \qquad r_n'=G(R_n,R_n'), \qquad (5.2)$$

where $F(\cdot,\cdot)$ and $G(\cdot,\cdot)$ are functions that can be explicitly computed. Because the expressions are too complicated, we have decided not to write them down. Due to the fact that $\frac{\mathrm{d}}{\mathrm{d}t}F(R_n, R_n') = r_n' = G(R_n, R_n')$, it can be shown that

$$\big[(2n+\alpha+\beta+\gamma)(2n+\alpha+\beta+\gamma+1)+((2n+\alpha+\beta+\gamma+1)t \\ -2(2n+\alpha+\beta+\gamma)-1)R_n(t)-(t-1)R_n^2(t) \\ +t(t-1)R_n'(t)\big]\Phi(R_n, R_n', R_n'')=0, \tag{5.3}$$

where $\Phi(\cdot,\cdot,\cdot)$ is a functions that is explicitly known. Obviously, the above formula yields two differential equations. One is a first-order differential equation, actually a Riccati equation, whose solution is given by

$$R_n(t)=\frac{(2n+\alpha+\beta+\gamma+1)(1+\lambda(2n+\alpha+\beta+\gamma+1)(1-t)^{2n+\alpha+\beta+\gamma})}{1+\lambda(2n+\alpha+\beta+\gamma+1)(1-t)^{2n+\alpha+\beta+\gamma+1}}, \tag{5.4}$$

where λ is an integration constant. However, as $t\to-\infty$, it is easily seen that

$$R_n(t)\to\begin{cases}0, & \text{if}\quad \lambda\neq 0\\ 2n+\alpha+\beta+\gamma+1, & \text{if}\quad \lambda=0,\end{cases} \tag{5.5}$$

which violates the result $R_0(t)\sim\alpha+\beta+1$ in (3.41). So we discard this Riccati equation.

Finally, applying a suitable rescaling and displacement as given in (1.15), we obtain the Painlevé VI equation (1.16) from $\Phi(\cdot,\cdot,\cdot)=0$ in (5.3), and this completes the proof of our theorem. □

Acknowledgments

We thank the referees for providing constructive comments to improve the contents of this paper. The work of DD was partially supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China [Project No. CityU 100909].

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